

# A PATHOLOGICAL CONSTRUCTION FOR REAL FUNCTIONS WITH LARGE COLLECTIONS OF LEVEL SETS

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**ABSTRACT.** Consider all the level sets of a real function. We can group these level sets according to their Hausdorff dimensions. We show that the Hausdorff dimension of the collection of all level sets of a given Hausdorff dimension can be arbitrarily close to 1, even if the function is differentiable to some level. By definition of Hausdorff dimension it is clear, for any real function  $f(x)$  and any  $\alpha \in [0, 1]$ , that  $\dim_H \{ y : \dim_H(f^{-1}(y)) \geq \alpha \} \leq 1$ . What is surprising, and what we show, is that this is actually a sharp bound. That is,

$$\sup \{ \dim_H \{ y : \dim_H(f^{-1}(y)) = 1 \} : f \in C^k \} = 1,$$

for any  $k \in \mathbb{Z}_{\geq 0}$ .

## 1. PRELIMINARIES

For the purposes of this paper it will be sufficient to consider functions of the form

$$f : [0, 1] \rightarrow [0, 1].$$

Let  $y \in [0, 1]$  and consider the level set  $f^{-1}(y) \subseteq [0, 1]$ .

For any  $d \in [0, \infty)$ , this level set has a  $d$ -dimensional **Hausdorff content** given by

$$C_H^d(f^{-1}(y)) = \inf \left\{ \sum_i r_i^d : \text{there is a cover of } f^{-1}(y) \text{ by balls of radii } r_i > 0 \right\}.$$

Further,  $f^{-1}(y)$  has a **Hausdorff dimension** given by

$$\dim_H(f^{-1}(y)) = \inf \{ d \geq 0 : C_H^d(f^{-1}(y)) = 0 \}.$$

We are interested in all those  $y$  whose pre-images have positive Hausdorff dimension:

$$\{ y \in [0, 1] : \dim_H(f^{-1}(y)) > 0 \}.$$

More specifically though we are interested in the sets

$$\{ y \in [0, 1] : \dim_H(f^{-1}(y)) \geq \alpha \},$$

where  $0 \leq \alpha \leq 1$ .

We wish to find functions,  $f(x)$ , that maximize the Hausdorff dimension of this set.

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*Date:* August 29, 2016.

*2000 Mathematics Subject Classification.* Primary: 26A06, 26A18, 28A78, Secondary: 37E05, 28A80.

*Key words and phrases.* Collections of level sets, Hausdorff dimension, Real functions.

**Definition 1.1.** Let  $0 \leq \alpha \leq 1$ . Define

$$I_\alpha(f) = \{ y \in [0, 1] : \dim_H(f^{-1}(y)) \geq \alpha \}.$$

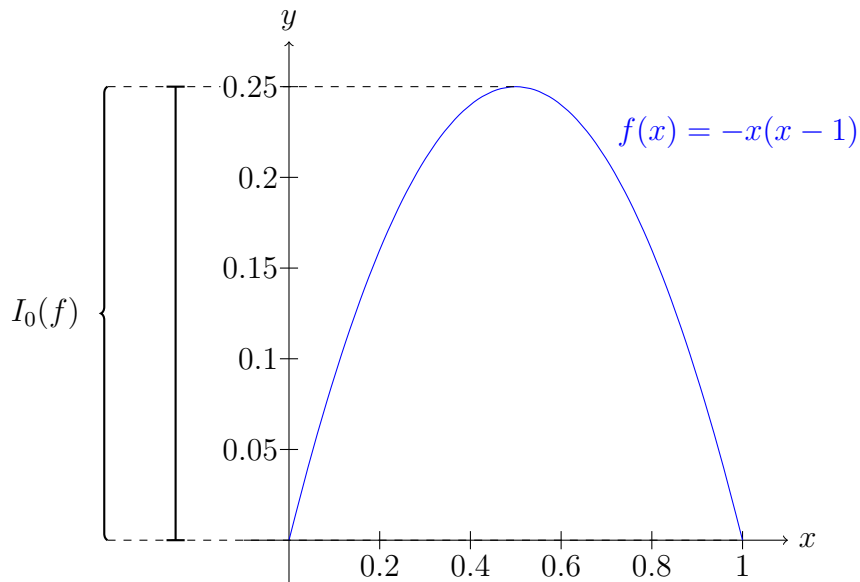
Note: Trivially, for any function  $f(x)$ , we have  $I_0(f) = \text{Range}(f)$ .

## 2. EXAMPLES

*Example 2.1 (Trivial Example).*

Consider the graph of the function

$$\begin{aligned} f : [0, 1] &\longrightarrow [0, 1] \\ x &\longmapsto -x(x - 1). \end{aligned}$$



As expected, in this case  $I_0(f) = [0, 0.25]$ .

Note that the pre-image of each point in the range of  $f(x)$  is at most finite. Thus the pre-image of each point has trivial Hausdorff dimension. Hence

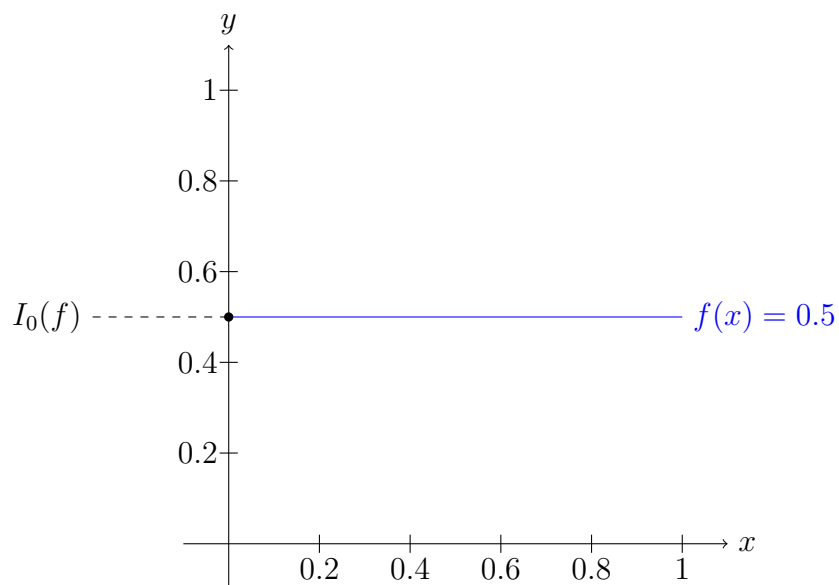
$$I_\alpha(f) = \emptyset \quad \text{and} \quad \dim_H I_\alpha(f) = 0,$$

for all  $\alpha > 0$ .

*Example 2.2 (Another Trivial Example).*

Consider any constant function. For example:

$$f : [0, 1] \longrightarrow [0, 1], \quad x \longmapsto 0.5.$$



In this case the only non-trivial pre-image is  $f^{-1}(0.5) = [0, 1]$ .

The unit interval has Hausdorff dimension 1, and so

$$I_\alpha(f) = \{0.5\} \quad \text{and} \quad \dim_H I_\alpha(f) = 0,$$

for all  $0 \leq \alpha \leq 1$ .

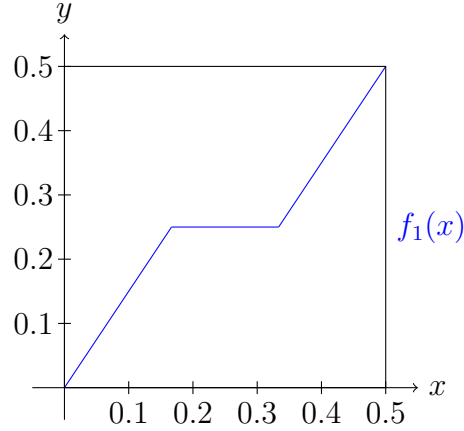
The next question is: How large can we make  $I_\alpha(f)$ , for  $\alpha > 0$ , while preserving continuity or even differentiability?

The next example shows that we can construct a continuous function  $f(x)$  such that  $I_1(f)$  is infinite.

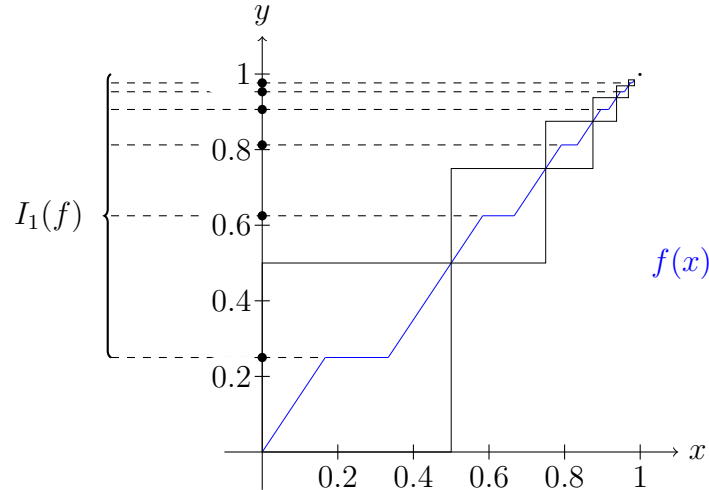
*Example 2.3. (Non-Trivial  $I_\alpha(f)$ )*

Consider the function

$$f_1 : [0, 0.5] \longrightarrow [0, 1], \quad x \longmapsto \begin{cases} \frac{3}{2}x & \text{if } x \in [0, \frac{1}{6}] \\ \frac{1}{4} & \text{if } x \in [\frac{1}{6}, \frac{1}{3}] \\ \frac{3}{2}x - \frac{1}{4} & \text{if } x \in [\frac{1}{3}, \frac{1}{2}] \end{cases}$$



Take this function and make scaled copies of it with dimensions  $\frac{1}{2^k} \times \frac{1}{2^k}$ . Then graph these scaled functions end-to-end so that the bottom left coordinate of the  $k$ -th graph coincides with the point  $(1 - \frac{1}{2^{k-1}}, 1 - \frac{1}{2^{k-1}})$ .



This gives us a continuous (although not differentiable) function  $f : [0, 1] \longrightarrow [0, 1]$  such that

$$I_\alpha(f) = \left\{ \frac{1}{4}, \frac{5}{8}, \dots, \frac{2^i - \frac{3}{2}}{2^i}, \dots \right\} \quad \text{and} \quad \dim_H I_\alpha(f) = 0,$$

for all  $0 < \alpha \leq 1$ .

### 3. MAIN THEOREM

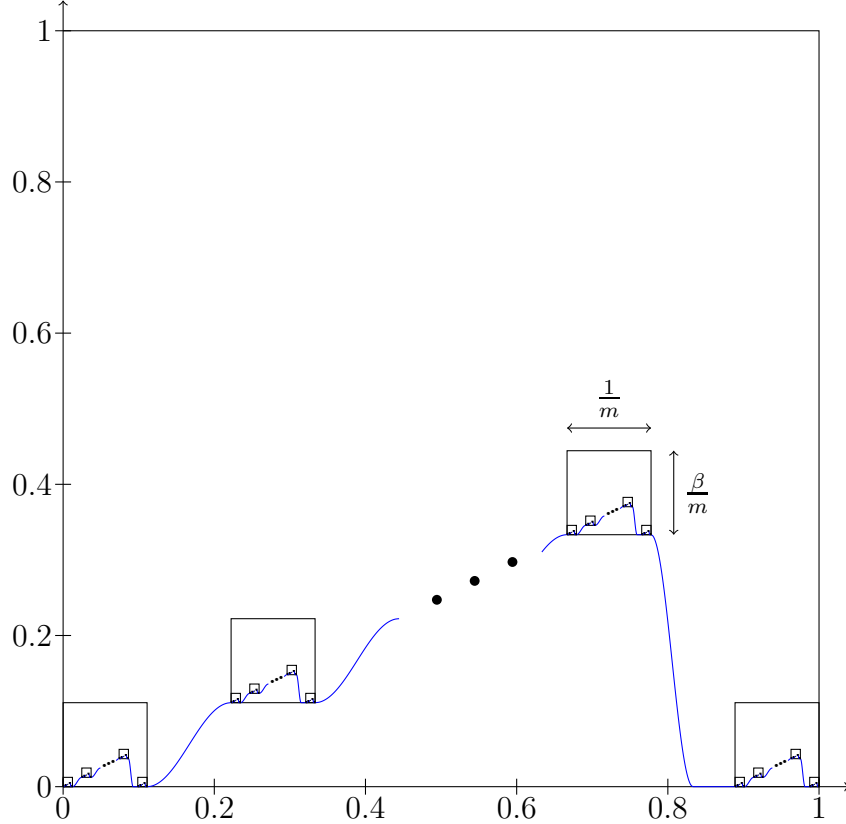
In this paper we show the following very counterintuitive result:

We can make  $\dim_H I_\alpha(f) \leq 1$  arbitrarily close to 1, for all  $0 \leq \alpha \leq 1$ , while still maintaining the continuity and even differentiability of  $f(x)$ .

**Theorem 3.1.** *For any  $k \in \mathbb{Z}_{\geq 0}$  and any  $0 \leq \alpha \leq 1$  we have*

$$\sup \{ \dim_H(I_\alpha(f)) : f \in C^k \} = 1.$$

*Example 3.2. (Main Function)* Consider the following iteratively defined function.



Let  $k \in \mathbb{Z}_{\geq 0}$  and  $\beta < 1$ .

Let  $n$  refer to the level of iteration we are considering at a given time.

Let  $b$  be the number of boxes in the initial iteration level ( $n = 1$ ), and let  $m$  be the total number of solid curves and boxes in the initial iteration. We shall choose  $b = \frac{m+1}{2}$ .

Note: This forces  $m$  to be an odd natural number.

**Construction at iteration level  $n = 1$ .** We begin with  $b$  boxes of dimension  $\frac{1}{m} \times \frac{\beta}{m}$  arranged in the unit square so that the first  $b - 1$  boxes form a diagonal with bottom left corners having coordinates  $(\frac{2i}{m}, \frac{i}{m})$ , for  $0 \leq i \leq b - 2$ . The remaining box then has its bottom left corner placed at  $(\frac{2b-2}{m}, 0)$ .

To connect the first  $b - 1$  boxes we use smooth curves beginning at the bottom right-hand corner of one box and ending at the bottom left-hand corner of the next box. We choose these curves,  $g_{k1}(x)$ , to be translations of the solution to

$$\frac{dg_{k1}}{dx} = C(mx)^k(1 - mx)^k, \quad g_{k1}(0) = 0, \quad g_{k1}\left(\frac{1}{m}\right) = \frac{\beta}{m},$$

on the interval  $[0, \frac{1}{m}]$ , for some constant  $C$ . This constant is given in [2].

Note: Any suitable flat function would work here, all we require is a  $C^k$  function on a closed interval with trivial first  $k$ -derivatives at both ends.

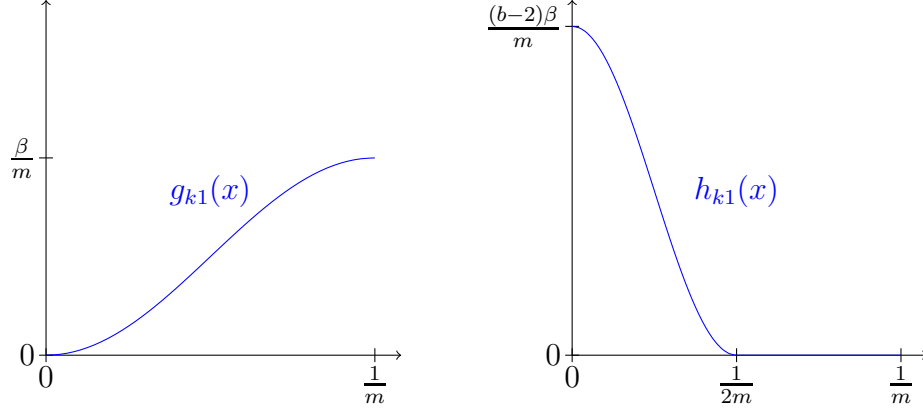
Solving the above ODE gives us the following connecting curves

$$\begin{aligned} g_{k1}(x) &= \frac{\beta}{m} \frac{1}{\sum_{i=0}^k \binom{k}{i} \frac{(-1)^i}{k+1+i}} \sum_{i=0}^k \binom{k}{i} \frac{(-1)^i}{k+1+i} (mx)^{k+1+i} \\ &= \frac{\beta}{m} \frac{(2k+1)!}{(k!)^2} \sum_{i=0}^k \binom{k}{i} \frac{(-1)^i}{k+1+i} (mx)^{k+1+i}. \end{aligned}$$

To join the penultimate box to the final box we use a translation of the previous curve combined with a reflection and scaling:

$$h_{k1}(x) = \frac{\beta}{m} (b-2) \frac{(2k+1)!}{(k!)^2} \sum_{i=0}^k \binom{k}{i} \frac{(-1)^i}{k+1+i} (1 - 2mx)^{k+1+i},$$

for  $0 \leq x \leq \frac{1}{m}$ , and  $h_{k1}(x) = 0$  for  $\frac{1}{2m} \leq x \leq \frac{1}{m}$ .



This gives us the first iteration:  $n = 1$ .

For the next iteration,  $n = 2$ , we take the  $b$  boxes of dimension  $\frac{1}{m} \times \frac{\beta}{m}$ , and into each of these boxes we identically construct a new collection of boxes and curves similar to those in iteration  $n = 1$ , with the exception that the new boxes have dimension  $\frac{1}{m^2} \times \frac{\beta^2}{m^2}$  and the new curves are all appropriately scaled so that they are all translations of

$$g_{k2}(x) = \frac{\beta}{m} g_{k1}(mx) \quad \text{and} \quad h_{k2}(x) = \frac{\beta}{m} h_{k1}(mx).$$

We then repeat this process ad infinitum, for each iteration  $n$ .

This gives us our function  $f(x) : [0, 1] \rightarrow [0, 1]$ .

*Claim 3.3.*  $f(x) \in C^k([0, 1])$ .

*Proof.* The domain of  $f(x)$  can be broken in to two groups: interior points on which the solid curves are defined and boundary points at the left and right endpoints of some box.

It is clear that  $f \in C^k$  for any interior point on which a solid curve is defined. It remains to establish that  $f \in C^k$  at the endpoints of the boxes. More specifically, it remains to establish that  $f(x)$  is  $k$ -times differentiable from the left for right-hand endpoints, and from the right for left-hand endpoints. We prove this by induction on order of differentiation  $1 \leq j \leq k$ .

Let  $x = e$  be any left endpoint of some box from our construction process.

**Case:**  $j = 1$ .

Let  $(e_i)_{i \in \mathbb{N}}$  be any sequence of points, for which we have defined right-hand derivatives, that converge from the right to  $e$ .

By construction, for any given  $i \in \mathbb{N}$  there exists  $N(i) \in \mathbb{N}$  that tells us the level of the iterative process at which  $f(e_i)$  was defined. Since  $\lim_{i \rightarrow \infty} e_i = e$  it follows that  $\lim_{i \rightarrow \infty} N(i) = \infty$ .

If  $f(e_i)$  is defined in the  $N(i)$ -th level of the iterative process then

$$\left(\frac{1}{m}\right)^{N(i)} \leq |e - e_i| < \left(\frac{1}{m}\right)^{N(i)-1} \quad \text{and} \quad |f(e) - f(e_i)| < \left(\frac{\beta}{m}\right)^{N(i)-1}.$$

Hence

$$\frac{|f(e) - f(e_i)|}{|e - e_i|} < \beta^{N(i)-1} m.$$

By definition,  $\beta < 1$ , and thus

$$\partial_+ f(e) = \lim_{i \rightarrow \infty} \frac{|f(e) - f(e_i)|}{|e - e_i|} \leq \lim_{i \rightarrow \infty} \beta^{N(i)-1} m = 0 = \partial_- f(e),$$

the last equality comes from our choice of the solid curves.

This argument is virtually identical for right endpoints. Therefore  $f(x) \in C^1$  and  $f^{(1)}(e) = 0$ .

**Case**  $j = l \leq k$ .

Assume that  $f^{(1)}(e) = \dots = f^{(l-1)}(e) = 0$  for some left endpoint,  $e$ , of a box. Again, let  $(e_i)_{i \in \mathbb{N}}$  be any sequence of points, for which we have defined right-hand derivatives, that converge from the right to  $e$ .

As above, there exists  $N(i) \in \mathbb{N}$  telling us the level of the iterative process at which  $f(e_i)$  is defined.

Consider  $|f^{(l-1)}(e) - f^{(l-1)}(e_i)| = |f^{(l-1)}(e_i)|$ . When defining the solid curve on  $x = e_i$  we used a translation of one of the polynomials  $g_{kN(i)}(x)$  or  $h_{kN(i)}(x)$ . Thus

$$|f^{(l-1)}(e) - f^{(l-1)}(e_i)| = |f^{(l-1)}(e_i)| = \left| g_{kN(i)}^{(l-1)}(x) \right| \quad \text{or} \quad \left| h_{kN(i)}^{(l-1)}(x) \right|.$$

In our construction we chose that

$$g_{kN(i)}^{(1)}(x) = \frac{\beta^{N(i)}}{m^{N(i)}} \frac{(2k+1)!}{(k!)^2} (m^{N(i)}x)^k (1 - m^{N(i)}x)^k$$

on  $[0, \frac{1}{m^{N(i)}}]$  and

$$h_{kN(i)}^{(1)}(x) = \frac{d}{dx} \left( g_{kN(i)} \left( \frac{1}{m^{N(i)}} - 2x \right) \right) = -2g_{kN(i)}^{(1)} \left( \frac{1}{m^{N(i)}} - 2x \right),$$

on  $[0, \frac{1}{2m^{N(i)}}]$  and  $h_{kN(i)}^{(1)}(x) = 0$  on  $[\frac{1}{2m^{N(i)}}, \frac{1}{m^{N(i)}}]$ .

Hence

$$g_{kN(i)}^{(l-1)}(x) = \frac{\beta^{N(i)}}{m^{N(i)}} (m^{N(i)}x)^{k+2-l} (1 - m^{N(i)}x)^{k+2-l} p_{k,l-1,i}(m^{N(i)}x),$$

for some polynomial  $p_{k,l-1,i}(m^{N(i)}x)$  of order  $l-2$  defined on  $[0, \frac{1}{m^{N(i)}}]$ . Also

$$h_{kN(i)}^{(l-1)}(x) = (-2)^{(l-1)} g_{kN(i)}^{(l-1)} \left( \frac{1}{m^{N(i)}} - 2x \right),$$

on  $[0, \frac{1}{2m^{N(i)}}]$  and  $h_{kN(i)}^{(l-1)}(x) = 0$  on  $[\frac{1}{2m^{N(i)}}, \frac{1}{m^{N(i)}}]$ .

This tells us three things:

1. The first  $k$  right-derivatives of the solid curves at their left end-points are equally 0,
2. The first  $k$  left-derivatives of the solid curves at their right end-points are equally 0,
3. Since  $p_{k,l-1,i}(m^{N(i)}x)$  is a polynomial defined on  $[0, \frac{1}{m^{N(i)}}]$  it must be bounded by some constant  $c(k, l)$  only depending on  $k$  and  $l$ . Therefore

$$\begin{aligned} |f^{(l-1)}(e) - f^{(l-1)}(e_i)| &\leq \max \left\{ \left| g_{kN(i)}^{(l-1)}(x) \right|, \left| h_{kN(i)}^{(l-1)}(x) \right| \right\} \\ &\leq \frac{\beta^{N(i)}}{m^{N(i)}} C(k, l), \end{aligned}$$

where  $C(k, l)$  is some constant depending on  $k$  and  $l$ .

Now, as in the initial case, we have that if  $f(e_i)$  is defined in the  $N(i)$ -th level of the iterative process then

$$\left( \frac{1}{m} \right)^{N(i)} \leq |e - e_i| < \left( \frac{1}{m} \right)^{N(i)-1} \quad \text{and} \quad |f^{(l-1)}(e) - f^{(l-1)}(e_i)| < \left( \frac{\beta}{m} \right)^{N(i)} C(k, l).$$

Hence

$$\frac{|f^{(l-1)}(e) - f^{(l-1)}(e_i)|}{|e - e_i|} \leq \beta^{N(i)} C(k, l).$$

Taking the limit as  $i \rightarrow \infty$ :

$$\partial_+ f^{(l-1)}(e) = \lim_{i \rightarrow \infty} \frac{|f(e) - f(e_i)|}{|e - e_i|} \leq \lim_{i \rightarrow \infty} \beta^{N(i)} C(k, l) = 0 = \partial_- f^{(l-1)}(e).$$



The argument is virtually identical for right endpoints. Thus  $f(x) \in C^l$ . This gives us the inductive step.

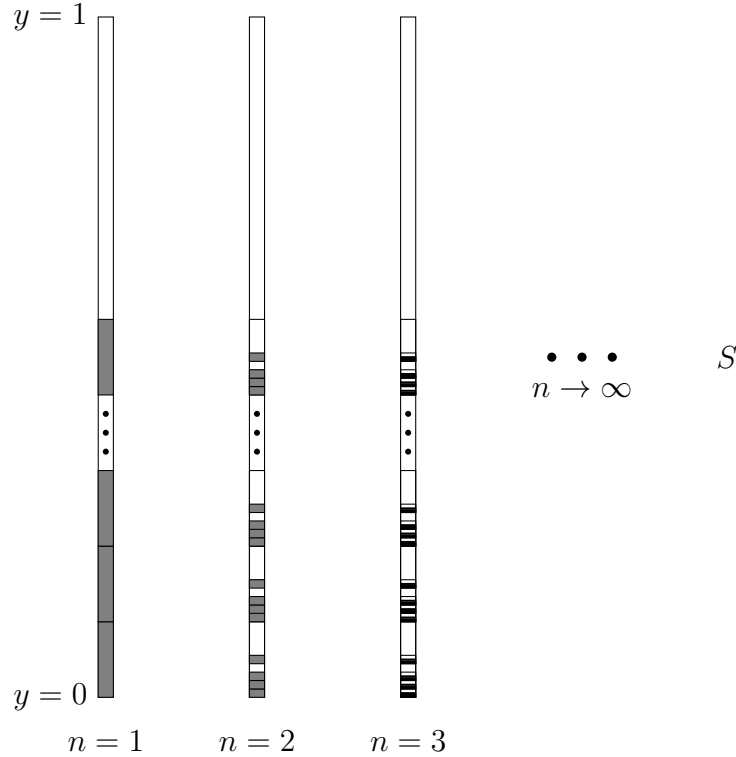
Therefore, by strong induction,  $f(x) \in C^k([0, 1])$ .

□

*Claim 3.4.*  $\dim_H I_1 = \frac{\log(b-1)}{\log(2b+1)-\log(\beta)}$ .

*Proof.* In each level of the iteration we added flat sections of curves. These flat sections mean that  $f(x)$  has points in its range whose pre-images have Hausdorff dimension 1.

We want to calculate the Hausdorff dimension of the collection of all these points in the range of  $f(x)$ , which is equivalent to calculating the Hausdorff dimension of the intersection of all the boxes in the range. Let us denote this set by  $S$ .



Set  $d = \frac{\log(b-1)}{\log(2b+1)-\log(\beta)}$ . We first prove that  $\dim_H(S) \leq d$ . Suppose  $\gamma > d$ . The iterative process used to construct  $f(x)$  gives us a sequence of coverings of  $S$ . At level  $n = 1$  we can cover  $S$  by  $b - 1$  intervals of length  $\frac{\beta}{m}$ . At level  $n = 2$  we can cover  $S$  by  $(b - 1)^2$  intervals of length  $\left(\frac{\beta}{m}\right)^2$ . After  $n$  iterations we can cover  $S$  by  $(b - 1)^n$  intervals of length  $\left(\frac{\beta}{m}\right)^n$ . The  $\gamma$ -total length of the  $n$ -th cover of  $S$  is then  $(b - 1)^n \left(\frac{\beta}{m}\right)^{\gamma n}$ .

If we take the limit of this as  $n \rightarrow \infty$  we get

$$\lim_{n \rightarrow \infty} (b - 1)^n \left(\frac{\beta}{m}\right)^{\gamma n} = \lim_{n \rightarrow \infty} \exp [n (\log(b - 1) - \gamma (\log(m) - \log(\beta)))] = 0.$$

Therefore  $C_H^\gamma(S) = 0$  and  $\dim_H(S) \leq d = \frac{\log(b-1)}{\log(2b+1)-\log(\beta)}$ .

For the other direction we will show that  $C_H^d(S) > 0$ .

Let  $(S_i)_{i \in \mathbb{N}}$  be a countable cover of  $S$ .

By compactness [3], given any  $\varepsilon > 0$ , there exist a finite collection of open intervals  $D_1, \dots, D_l$  such that  $\cup_{i=1}^{\infty} S_i \subseteq \cup_{j=1}^l D_j$  and

$$\sum_{j=1}^l |D_j|^\alpha < \sum_{i=1}^{\infty} |S_i|^\alpha + \varepsilon.$$

Let us choose  $n$  such that

$$\left(\frac{\beta}{m}\right)^n \leq \min \{|D_j| : j = 1, \dots, l\}.$$

For  $i = 1, \dots, n$  define

$$M_i = \# \left\{ D_j : \left(\frac{\beta}{m}\right)^i \leq |D_j| < \left(\frac{\beta}{m}\right)^{i-1} \right\}.$$

It follows that

$$\sum_{j=1}^l |D_j|^\alpha \geq \sum_{j=1}^n M_j \left(\frac{\beta}{m}\right)^{j\alpha} = \sum_{j=1}^n M_j \left(\frac{1}{b-1}\right)^j.$$

Consider any  $D_j$ . There must exist some  $i$  such that  $\left(\frac{\beta}{m}\right)^i \leq |D_j| < \left(\frac{\beta}{m}\right)^{i-1}$ . Thus  $D_j$  can intersect at most 2 of the  $(b-1)^i$  intervals obtained in the  $i$ -th level of the iterative process. Each of these intervals produces  $(b-1)^{n-i}$  sub-intervals at the  $n$ -th level of the iterative process, hence  $D_j$  contains at most  $2(b-1)^{n-i}$  intervals from the  $n$ -th level of the construction process. In total, the  $n$ -th step of the construction process has  $(b-1)^n$  intervals. Therefore

$$(b-1)^n \leq \sum_{i=1}^l 2M_i(b-1)^{n-i} \Rightarrow \frac{1}{2} \leq \sum_{i=1}^l \frac{M_i}{(b-1)^i}.$$

Combining this with the above equation gives:

$$\frac{1}{2} \leq \sum_{j=1}^l |D_j|^d < \sum_{i=1}^{\infty} |S_i|^d + \varepsilon.$$

Let  $\varepsilon = \frac{1}{4}$ . Then

$$\frac{1}{4} < \sum_{i=1}^{\infty} |S_i|^d.$$

Therefore  $\sum_{i=1}^{\infty} |S_i|^d$  is bounded below and hence

$$\dim_H(S) \geq d = \frac{\log(b-1)}{\log(2b+1) - \log(\beta)}.$$

□

Using the previous claim and letting  $b \rightarrow \infty$ , L'Hôpital's Rule tells us that:

$$\lim_{b \rightarrow \infty} \frac{\log(b-1)}{\log(2b+1) - \log(\beta)} = \lim_{b \rightarrow \infty} \left[ \frac{\frac{1}{b-1}}{\frac{2}{2b+1}} \right] = \lim_{b \rightarrow \infty} \left[ 1 + \frac{3}{2b-2} \right] = 1.$$

Therefore 1 is indeed a sharp bound for  $\dim_H I_1(f) \leq 1$ .

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